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An Abstract Averaging Theorem

THOMAS G. KURTZ*

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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For each $t \geqslant 0$, let $A(t)$ generate a contraction semigroup on a Banach space L. Suppose the solution of $u_t = \epsilon A(t)u$ is given by an evolution operator $V_\epsilon(t, s)$. Conditions are given under which $V_\epsilon((t+s)/\epsilon, s/\epsilon)$ converges strongly as $\epsilon \to 0$ to a semigroup $T(t)$ generated by the closure of $\bar{A}f = \lim_{T\to\infty}(1/T) \int_0^T A(t) f\, dt$.

This result is applied to the following situation: Let B generate a contraction group $S(t)$ and the closure of $\epsilon A + B$ generate a contraction semigroup $S_\epsilon(t)$. Conditions are given under which $S(-t/\epsilon)\, S_\epsilon(t/\epsilon)$ converges strongly to a semigroup generated by the closure of $\bar{A}f \equiv \lim_{T\to\infty}(1/T) \int S(-t)\, AS(t) f\, dt$. This work was motivated by and generalizes a result of Pinsky and Ellis for the linearized Boltzmann Equation.

Formally we are interested in the behavior as ϵ goes to zero of solutions of

$$(d/dt)\, v = \epsilon A(t) v, \tag{1}$$

where $v(t)$ is in a Banach space L and for each t, $A(t)$ is the generator of a contraction semigroup.

After a change of variable (1) becomes

$$(d/dt)\, u = A(t/\epsilon) u. \tag{2}$$

We study this equation by considering the evolution operator $U_\epsilon(t, s)$ that corresponds to (2) at least in a weak sense.

THEOREM (3). *For every $t \geqslant 0$ let $A(t)$ be the generator of a strongly continuous contraction semigroup on a Banach space L. For each $\epsilon > 0$, let $U_\epsilon(t, s)$, $t > s$, be a family of contraction operators satisfying $U_\epsilon(t, s)\, U_\epsilon(s, r) = U_\epsilon(t, r)$, and suppose $U_\epsilon(t, s)$ is strongly continuous as a function of t and s.*

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Let D be the collection of $f \in L$ with the following properties:

$$f \in \mathscr{D}(A(t)) \qquad \text{for every} \quad t \geqslant 0; \tag{4}$$

$$U_\epsilon(t, s) f = f + \int_s^t U_\epsilon(t, z)\, A(z/\epsilon) f\, dz \qquad \text{for all} \quad t \geqslant s \geqslant 0; \tag{5}$$

$$\sup_{t \geqslant 0} \| A(t) f \| < \infty; \tag{6}$$

$$\lim_{T \to \infty} (1/T) \int_0^T A(t) f\, dt \equiv \bar{A} f \text{ exists}; \tag{7}$$

for every $\eta > 0$ there is a compact set $K_\eta \subset L$ such that

$$\overline{\lim_{T \to \infty}}\, (1/T) \sup_s m\{t\colon s \leqslant t \leqslant s + T,\ A(t) f \notin K_\eta\} \leqslant \eta. \tag{8}$$

(Without loss of generality we can assume $\bar{A} f \in K_\eta$.)

If the closure of $\bar{A}$ generates a contraction semigroup $T(t)$ on L (i.e., if D and $\mathscr{R}(\lambda - \bar{A})$, some $\lambda > 0$, are dense in L) then

$$\lim_{\epsilon \to 0} \sup_{0 \leqslant s, t \leqslant k} \| U_\epsilon(t + s, s) f - T(t) f \| = 0 \tag{9}$$

for every $f \in L$ and every $k > 0$.

Remark. Condition (8) can be replaced by

$$\sup_{0 \leqslant s \leqslant t} \| A(t)\, A(s) f \| < \infty. \tag{8'}$$

Condition (8) will be satisfied, for example, if $A(t)$ is a periodic function of t, or if L is $C(R^r)$, f is twice continuously differentiable with compact support, the $A(t)$ are second order elliptic partial differential operators

$$A(t) f = \sum_{i,j} a_{ij}(x, t)\, \partial_i \partial_j f + \sum_i b_i(x, t)\, \partial_i f,$$

and

$$\{a_{ij}(\cdot, t), \qquad b_i(\cdot, t)\colon 1 \leqslant i, \quad j \leqslant r \quad t \geqslant 0\}$$

is a bounded uniformly equicontinuous family of functions.

If we strengthen (7) to

$$\lim_{T \to \infty} \sup_S \left\| (1/T) \int_S^{S+T} A(t) f\, dt - \bar{A} f \right\| = 0, \tag{7'}$$

then the convergence in (9) is uniform for $0 \leqslant S < \infty$.

Theorem (3) also holds for families of operators $U_\epsilon(s, t)$, $s \leqslant t$ satisfying $U_\epsilon(r, s)\, U_\epsilon(s, t) = U_\epsilon(r, t)$. In this case (5) is replaced by

$$U_\epsilon(s, t) f = f + \int_s^t U_\epsilon(s, z)\, A(z/\epsilon) f\, dz, \tag{5'}$$

condition (8′) becomes

$$\sup_{0 \leqslant s \leqslant t} \| A(s)\, A(t) f \| < \infty, \tag{8''}$$

and the conclusion becomes

$$\lim_{\epsilon \to 0} \sup_{0 \leqslant s, t \leqslant k} \| U_\epsilon(s, t + s) f - T(t) f \| = 0. \tag{9'}$$

The primary motivation behind the development of Theorem (3) is the following result which was suggested by work of Ellis and Pinsky [1, 2].

THEOREM (10). *Suppose B generates a contraction group $S(t)$, A generates a contraction semigroup, and the closure of $\epsilon A + B$ generates a contraction semigroup $S_\epsilon(t)$ for each $\epsilon > 0$ (all on the Banach space L).*

Let $U_\epsilon(t, s) = S(-t/\epsilon)\, S_\epsilon((t - s)/\epsilon)\, S(s/\epsilon)$.
If $S(t) f \in \mathscr{D}(A)$ all $t \geqslant 0$ and $\sup_{t \geqslant 0} \| AS(t) f \| < \infty$, then

$$U_\epsilon(t, s) f - f = \int_s^t U_\epsilon(t, z)\, S(-z/\epsilon)\, AS(z/\epsilon) f\, dz. \tag{11}$$

Let D be the collection of f satisfying the following conditions:

$$S(t) f \in \mathscr{D}(A) \qquad \text{all} \quad t > 0; \tag{12}$$

$$\sup_{t \geqslant 0} \| S(-t)\, AS(t) f \| \equiv \sup_{t \geqslant 0} \| AS(t) f \| < \infty; \tag{13}$$

$$\lim_{T \to \infty} (1/T) \int_0^T S(-t)\, AS(t) f\, dt \equiv \bar{A} f \text{ exists}; \tag{14}$$

for every $\eta > 0$, there is a compact set $K_\eta \subset L$ such that

$$\overline{\lim_{T \to \infty}}\, (1/T) \sup_s m\{t: s \leqslant t \leqslant s + T,\ S(-t)\, AS(t) f \notin K_\eta\} \leqslant \eta. \tag{15}$$

If the closure of $\bar{A}$ generates a strongly continuous contraction semigroup $T(t)$ on L, then

$$\lim_{\epsilon \to 0} S(-t/\epsilon)\, S_\epsilon(t/\epsilon) f = T(t) f \qquad \text{for every } f \in L. \tag{16}$$

Furthermore $T(t)$ and $S(t')$ commute so $V_\epsilon(t)f \equiv S(t/\epsilon)\,T(t)f$ defines a semigroup and

$$\lim_{\epsilon\to 0} \| S_\epsilon(t/\epsilon)f - V_\epsilon(t)f \| = 0 \qquad \text{for every } f \in L.$$

Remark. As in Theorem (3), condition 15 may be replaced by

$$\sup_{u,t\geqslant 0} \| AS(u)\,AS(t)f \| < \infty. \tag{15'}$$

Theorem (10) follows directly from Theorem (3). It is easy to check that $S(t')\,\bar{A}f = \bar{A}S(t')f$ and hence $S(t')$ and $T(t)$ commute.

Ellis and Pinsky [1, 2] prove (16) in the case where A and B are matrices and use their result to obtain a special case of (16) related to the linearized Boltzmann equation.

Kato [4] proves (16) in the case where A is a bounded operator using the perturbation expansion for $S_\epsilon(t)$. (See Kato [3, p. 495]). In this case conditions (12), (13), and (15′) are immediate. Kato also shows that (14) holds if $S(t)$ is a unitary group on Hilbert space and A is compact.

Define

$$Pf \equiv \lim_{T\to\infty} (1/T) \int_0^T S(t)f\,dt \tag{17}$$

where the limit exists. If the closure of PA generates a contraction semigroup $U(t)$ on the range of P, $\mathscr{R}(P)$, then [7, Theorem (2.1)] implies

$$\lim_{\epsilon\to 0} S_\epsilon(t/\epsilon)f = U(t)f \qquad \text{for} \quad f \in \mathscr{R}(P). \tag{18}$$

It follows that $U(t) = T(t)f$ for $f \in \mathscr{R}(P)$. Consequently, Theorem (10) is an extension of (18), under the hypothesis that B generates a group rather than a semigroup. See Papanicolaou and Kohler [10] for an extension of the results in [7] in a different direction.

Proof of Theorem (3). The proof is essentially the same as the proof of [6, Theorem (2.1)]. We give it here because the modifications are not completely straightforward and the probabilistic setting of the result in [6] obscures the relationship to the current work.

We begin by extending the definition of $U_\epsilon(t, s)$ to all values $t \geqslant s$ as follows:

Let $A(s)$ be the closure of $\bar{A}$ for $s < 0$ and define

$$U_\epsilon(t, s) = T(t - s) \qquad \text{for} \quad 0 \geqslant t \geqslant s$$

and

$$U_\epsilon(t, s) = U_\epsilon(t, 0)\,T(-s) \qquad \text{for} \quad t \geqslant 0 \geqslant s.$$

Note that (5) holds for all $t \geqslant s$ and

$$\lim_{T\to\infty} \sup_{-\infty < S \leqslant kT} \left\| (1/T) \int_S^{S+T} A(t) f \, dt - \bar{A} f \right\| = 0 \tag{19}$$

for every $f \in D$, and every $k > 0$.

We will need the following technical results:

The limit in (19) implies the existence for each $f \in D$ of a continuous, increasing function $\alpha(T)$ such that $\lim_{T\to\infty} \alpha(T) = \infty$, and

$$\lim_{T\to\infty} \sup_{-\infty < S \leqslant \alpha(T)T} \left\| (1/T) \int_S^{S+T} A(t) f \, dt - \bar{A} f \right\| = 0. \tag{20}$$

Note also that $f \in D$

$$\| U_\epsilon(s, s-t) f - f \| \leqslant t \sup_u \| A(u) f \|, \tag{21}$$

and since the bound on the right is independent of s and ϵ and D is dense in L,

$$\lim_{t\to 0} \sup_{\epsilon>0,s} \sup_{f\in K} \| U_\epsilon(s, s-t) f - f \| = 0, \tag{22}$$

for every compact subset $K \subset L$.

Let $\mathscr{L}$ be the strongly continuous functions $f\colon (-\infty, \infty) \to L$ with $\lim_{s\to\infty} f(s) = 0$ and $f(s) = f(0)$ for $s < 0$. Define $\|f(\cdot)\| = \sup \|f(s)\|_L$. Then

$$T_\epsilon(t) f(s) = U_\epsilon(s, s-t) f(s-t)$$

defines a strongly continuous semigroup on $\mathscr{L}$. The idea of the proof is to apply the semigroup convergence theorem given in [5] (see [6, Theorem (2.4)]) to $T_\epsilon(t)$. The notion of convergence we will use is the following: for $g_\epsilon(\cdot) \in \mathscr{L}$, define $\mathrm{LIM}\, g_\epsilon(\cdot) = g \in L$ if and only if $\sup \| g_\epsilon(\cdot)\| < \infty$ and $\lim_{\epsilon\to 0} \sup_{s\leqslant k} \| g_\epsilon(s) - g \| = 0$ for every $k > 0$. Note that ϵ corresponds to $1/\lambda$ in [6, Theorem (2.4)], and $\mathscr{L}$ corresponds to M_λ. Condition (2.5) of [6] follows by noting that $\lim_{\epsilon\to 0} \sup_{s\leqslant k+t} \| g_\epsilon(s)\| = 0$ implies $\lim_{\epsilon\to 0} \sup_{s\leqslant k} \| T_\epsilon(t) g_\epsilon(s)\| = 0$, and [6, (2.6)] then follows by the dominated convergence theorem.

Formally the generator for $T_\epsilon(t)$ is given by

$$A_\epsilon f(s) = A(s/\epsilon) f(s) - (d/ds) f(s).$$

Let $f \in D$ and $\gamma(s)$ be real-valued and continuous with $\gamma(s) = 1$ for $s \leqslant 0$ and $\lim_{s\to\infty} \gamma(s) = 0$. Then, letting $h(s) = \gamma(s) f$

$$g(s) = (1/\delta) \int_0^\delta \gamma(s-t)\, U_\epsilon(s, s-t) f \, dt = (1/\delta) \int_0^\delta T_\epsilon(t)\, h(s)\, dt \tag{23}$$

is in $\mathscr{D}(A_\epsilon)$ for $\delta > 0$ and

$$
\begin{aligned}
A_\epsilon g(s) &= \frac{\gamma(s-\delta)}{\delta} U_\epsilon(s, s-\delta) f - \frac{\gamma(s)}{\delta} f \\
&= \frac{\gamma(s-\delta) - \gamma(s)}{\delta} U_\epsilon(s, s-\delta) f \\
&\quad + \frac{\gamma(s)}{\delta} \int_{s-\delta}^{s} U_\epsilon(s, z)\, A(z/\epsilon) f\, dz \\
&= \frac{\gamma(s-\delta) - \gamma(s)}{\delta} U_\epsilon(s, s-\delta) f \\
&\quad + \frac{\gamma(s)}{\delta} \int_{s-\delta}^{s} (U_\epsilon(s, z) - I)\, A(z/\epsilon) f\, dz \\
&\quad + \gamma(s) \frac{\epsilon}{\delta} \int_{(s-\delta)/\epsilon}^{s/\epsilon} A(t) f\, dt.
\end{aligned}
$$

We will show that A_ϵ approximates $\bar{A}$ in the sense required by [6, Theorem (2.4)]. Let $\delta_\epsilon = \inf\{\delta: \delta > (\alpha(\delta/\epsilon))^{-1/2}\}$. Let γ_ϵ be non-increasing, continuously differentiable functions satisfying $\gamma_\epsilon(s) = 1$ for $s \leqslant 0$, $\lim_{s\to\infty} \gamma_\epsilon(s) = 0$, $\lim_{\epsilon\to 0} \gamma_\epsilon(s) = 1$ and $\lim_{\epsilon\to 0} \sup |\gamma_\epsilon'(s)| = 0$.

For $f \in D$ define

$$
g_\epsilon(s) = (1/\delta_\epsilon) \int_0^{\delta_\epsilon} \gamma_\epsilon(s-t)\, U_\epsilon(s, s-t) f\, dt.
$$

We observe that $\mathrm{LIM}\, g_\epsilon(s) = f$ and claim that $\mathrm{LIM}\, A_\epsilon g_\epsilon(s) = \bar{A} f$.

Substituting $\gamma_\epsilon(s)$ for $\gamma(s)$ and δ_ϵ for δ in (24), the first term on the right goes to zero since $\gamma_\epsilon'(s)$ goes to zero uniformly in s. The second term on the right is bounded by

$$
\begin{aligned}
&\sup_{t \leqslant \delta_\epsilon} \sup_{\epsilon > 0, s} \sup_{g \in K_\eta} \| U_\epsilon(s, s-t) g - g \| \\
&\quad + (\epsilon/\delta_\epsilon)\, m\{t: s/\epsilon - \delta_\epsilon/\epsilon \leqslant t \leqslant s/\epsilon, A(t) f \notin K_\eta\}\, 2 \sup_u \| A(u) f \|. \qquad (25)
\end{aligned}
$$

Consequently, (8) and (22) imply this term goes to zero uniformly in s (δ_ϵ/ϵ plays the role of T in (8)). (We could apply (8′) here instead of 8 and (22).)

The definition of δ_ϵ implies $s/\epsilon \leqslant \alpha(\delta_\epsilon/\epsilon)\, \delta_\epsilon/\epsilon$ for ϵ sufficiently small, and it follows from (20) that the third term on the right in (24) converges to $\bar{A} f$ uniformly in $s \leqslant k$.

The semigroup convergence theorem implies $\mathrm{LIM}\; T_\epsilon(t) f_\epsilon(\cdot) = T(t) f$ whenever $\mathrm{LIM}\, f_\epsilon(\cdot) = f$. In particular let $f_\epsilon(s) = \gamma_\epsilon(s) f$. Then

$$
\lim_{\epsilon \to 0} \sup_{s \leqslant k} \| U_\epsilon(s, s-t)\, \gamma_\epsilon(s-t) f - T(t) f \| = 0,
$$

for every k, which in turn implies

$$\lim_{\epsilon \to 0} \sup_{s \leqslant k} \| U_\epsilon(s + t, s) f - T(t) f \| = 0.$$

Uniformity of convergence for $0 \leqslant t \leqslant k$ follows from the uniform equicontinuity of $U_\epsilon(s + t, s)f$ as a function of t for all ϵ and s.

EXAMPLES. We will apply a modification of Theorem (3) to obtain a version of the classical averaging result for ordinary differential equations. (See [9].)

THEOREM (2.6). *Let $F(x, t)$ be continuous from $\mathbb{R}^r \times (0, \infty)$ into $\mathbb{R}^r$ and satisfy*

$$|F(x, t) - F(y, t)| \leqslant M |x - y| \tag{27}$$

for all $x, y \in \mathbb{R}^r$ and $t \geqslant 0$, and

$$\sup_{t \geqslant 0} |F(0, t)| < \infty. \tag{28}$$

Let $X_\epsilon(x, s, t)$ $0 \leqslant s \leqslant t$ satisfy

$$\begin{aligned} (d/dt) X_\epsilon(x, s, t) &= \epsilon F(X_\epsilon(x, s, t); t) \\ X_\epsilon(x, s, s) &= x. \end{aligned} \tag{29}$$

If

$$\lim_{T \to \infty} (1/T) \int_0^T F(x, t)\, dt = G(x), \tag{30}$$

where the convergence is uniform on compact subsets of $\mathbb{R}^r$ then

$$\lim_{\epsilon \to 0} X_\epsilon(x, s/\epsilon, s/\epsilon + t/\epsilon) = X(x, t) \qquad \textit{for} \quad s, \qquad t \geqslant 0, \tag{31}$$

where

$$\begin{aligned} (d/dt) X(x, t) &= G(X(x, t)) \\ X(x, 0) &= x \end{aligned} \tag{32}$$

and the convergence is uniform for x, s, and t in bounded sets.

Proof. Let L be the Banach space of continuous functions on $\mathbb{R}^r$ that vanish at infinity with the sup norm.

Define

$$A(t) f(x) = F(x, t) \cdot \operatorname{grad} f(x) \tag{33}$$

for f continuously differentiable with compact support. The closure of $A(t)$ generates a strongly continuous contraction semigroup on L.

Define

$$U_\epsilon(s, t) f(x) = f(X_\epsilon(x, s/\epsilon, t/\epsilon)).$$

Observe that

$$U_\epsilon(r, s)\, U_\epsilon(s, t) = U_\epsilon(r, t) \tag{34}$$

and

$$U_\epsilon(s, t) f = f + \int_s^t U_\epsilon(s, z)\, A(z/\epsilon) f\, dz. \tag{35}$$

This is the situation covered in the remark following Theorem (3). For f continuously differentiable with compact support, (4) is satisfied, (27) and (28) imply (6) and (8), and (30) implies (7). Since the closure of $G(x) \cdot \operatorname{grad} f(x)$ generates the semigroup $T(t) f(x) = f(X(x, t))$ (see [8, Theorem (1.1)]) we can conclude

$$\lim_{\epsilon\to 0} \sup_{t,s\leqslant k} \sup_{x} | f(X_\epsilon(x, s/\epsilon, (s + t)/\epsilon) - f(X(x, t))| = 0$$

for every f in L. This in turn implies (31).

The following result illustrates the use of Theorem (10).

THEOREM (31). *Let $X_\epsilon(t, x, y)$ and $Y_\epsilon(t, x, y)$ satisfy*

$$\begin{aligned} \dot X_\epsilon &= -Y_\epsilon + \epsilon G(X_\epsilon, Y_\epsilon) \\ \dot Y_\epsilon &= X_\epsilon + \epsilon H(X_\epsilon, Y_\epsilon), \\ X_\epsilon(0, x, y) &= x, \quad Y_\epsilon(0, x, y) = y. \end{aligned} \tag{37}$$

If G and H are Lipschitz continuous on $\mathbb{R}^2$ then

$$\lim_{\epsilon\to 0} X_\epsilon(t/\epsilon, X_0(-t/\epsilon, x, y),\ Y_0(-t/\epsilon, x, y)) = V(t, x, y)$$

and

$$\lim_{\epsilon\to 0} Y_\epsilon(t/\epsilon, X_0(-t/\epsilon, x, y),\ Y_0(-t/\epsilon, x, y)) = W(t, x, y)$$

where

$$\dot V = F_1(V, W)$$

and

$$\dot W = F_2(V, W)$$

$$(F_1 \textit{ and } F_2 \textit{ given below})$$

and the convergence is uniform for t, x, and y in bounded intervals.

Proof. Of course

$$X_0(t, x, y) = x \cos t - y \sin t$$

and

$$Y_0(t, x, y) = x \sin t + y \sin t.$$

This theorem is a straightforward application of Theorem (10) with

$$S_\epsilon(t) f(x, y) = f(X_\epsilon(t, x, y), Y_\epsilon(t, x, y))$$

and

$$S(t) f(x, y) = f(X_0(t, x, y), Y_0(t, x, y)).$$

If f is continuously differentiable with compact support then

$$\begin{aligned} Af(x, y) &= G(x, y) f_x(x, y) + H(x, y) f_y(x, y), \\ S(-t) AS(t) f(x, y) &= [\cos t\, G(x \cos t + y \sin t, -x \sin t + y \cos t) \\ &\quad - \sin t\, H(x \cos t + y \sin t, -x \sin t + y \cos t)] f_x(x, y) \\ &\quad + [\sin t\, G(x \cos t + y \sin t, -x \sin t + y \cos t) \\ &\quad + \cos t\, H(x \cos t + y \sin t, -x \sin t + y \cos t)] f_y(x, y). \end{aligned}$$

Periodicity implies

$$\bar{A}f = (1/2\pi) \int_0^{2\pi} S(-t) AS(t) f \, dt = F_1 f_x + F_2 f_y ,$$

which gives

$$\begin{aligned} F_1(x, y) = (1/2\pi) \int_0^{2\pi} &[\cos t\, G(x \cos t + y \sin t, -x \sin t + y \cos t) \\ &- \sin t\, H(x \cos t + y \sin t, -x \sin t + y \cos t)] \, dt. \end{aligned}$$

Similarly $F_2(x, y)$ is obtained.

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